PROJECT COMPLETION PROBABILITY AFTER CRASHING PERT/CPM NETWORK

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ABSTRACT: This paper analyzed a mathematical model for estimating the project completion probability after crashing PERT/CPM network. Here we reduced the pessimistic time of the activities on critical path to reduce the expected project completion time which increase the probability of the project completion on or before the scheduled time. Some additional cost is required for the activities along critical path to speed-up the project. The increment of the estimate cost decreases the pessimistic time of the activities along critical path and indirectly decreases the total expected project duration. The approach has been modelled in mathematical and algorithmic aspects and finally presented with a numerical example.

1. INTRODUCTION

CPM/PERT or Network Analysis as the technique is sometimes called, developed along two parallel streams, one industrial and the other military.

CPM was the discovery of M. R. Walker of E. I. Du Pont de Nemours & Co. and J. E. Kelly of Remington Rand, circa 1957. The computation was designed for the UNIVAC-I computer. The first test was made in 1958, when CPM was applied to the construction of a new chemical plant. In March 1959, the method was applied to maintenance shut-down at the Du Pont works in Louisville, Kentucky. Unproductive time was reduced from 125 to 93 hours.

PERT was devised in 1958 for the POLARIS missile program by the Program Evaluation Branch of the Special Projects office of the U. S. Navy, helped by the Lockheed Missile Systems division and the Consultant firm of Booz-Allen & Hamilton. The calculations were so arranged so that they could be carried out on the IBM Naval Ordinance Research Computer (NORC) at Dahlgren, Virginia.

Project crashing is a method for shortening the project duration by reducing the time of one or more of the critical project activities to less than its normal activity time. The object crashing is to reduce project duration while minimizing the cost of crashing.

Fulkerson (1961) and Premachandra (1992) reformulate the problem of crashing PERT network as a linear program, reducing the time required to find the optimal solution. Moore et al. (1978) reformulate the problem using goal programming. Goal programming is a modification of linear programming that can solve problems with multiple objectives. This allows goals in addition to cost minimization to be added to the problem. Foldes and Sourmis (1993) present a reformulation of crashing networks when the cost-time tradeoff is represented by non-linear, non-differentiable convex function. T Wei Peng at al (2010) analyzed numerical results of crashing CPM/PERT network to the increase of the probability or percentage of completing the project on or before the completion time by reducing pessimistic time of critical activities.

Crashing PERT networks was investigated by a number of researchers, such as Cho and Yum (1997), Ghaleb and Adnan (2001), Johnson and Schou (1990), Keefer and Verdini (1993), Saman (1991). There exist and example in Nicholas & Steyn (2008). that the probability that project will be finished before the expected duration are 1%-50%, while probability that project will be finished after the expected duration are 50%-99%.

The concept of crashing in CPM is applied to PERT networks in order to reduce the project duration of the project, and also to increase the probability of completing the project on or before completion time with the additional amounts of money that will be invested to activities on the critical path.

2. PERT FOR PROJECT COMPLETION PROBABILITY

PERT is a management technique to estimate the probability that a project will be finished on normal time. According to the traditional PERT technique the probability of a certain project meeting a specific schedule time can be described as follows:

$$Z = \frac{Due\ date - Expected\ date\ of\ project\ completion}{Project\ standard\ deviation}$$

$$Z = \frac{T_d - T_e}{\sigma_r}$$

Here, Z is the number of standard deviations of the due date or target date lies from the mean or expected date.

T_e is the normal expected time which is equal to the sum of normal expected times of activities on critical path. That means if $t_1, t_2, \ldots, t_k$ are the expected times of critical path activities, then

$$T_e = \sum_{i=1}^{k} t_i, i = 1, 2, \ldots \ldots \ldots, k.$$

T_d is the due date or targeted date of completion and σ_r is the project standard deviation which is written as

$$\sigma_r = \sqrt{project\ varience}$$
$$= \sqrt{\sum variance\ of\ activities\ on\ critical\ path}$$
$$= \sqrt{\sum_{i=1}^{k} v_i}$$

t_i and v_i for each activity are measured by the following formulae

$$t_i = \frac{a+4m+b}{6} \text{ and } v_i = \left(\frac{b-a}{6}\right)^2$$

The time estimates a, m, b are defined as follows:

Optimistic time (a): the minimum possible time required to accomplish a task, assuming everything proceeds better than is normally expected

Most likely time (m): the best estimate of the time required to accomplish a task, assuming everything proceeds as normal

Pessimistic time (b): the maximum possible time required to accomplish a task, assuming everything goes wrong.

3. PROJECT COMPLETION PROBABILITY TO MEET THE DESIRED DURATION

Concept of project crashing requires the investment of extra budget to minimize the duration so that the project can meet the targeted date. The delay of the activities on critical path makes the delay of the whole project. So, if we can finish the critical path activities before their normal estimated times, the project will be finished on or before the targeted time. For this, the extra budget will be invested for the activities along the critical path. The increasing of budget in critical path activities will minimize the total project duration and will increase the probability of the project completion on or before the targeted date.
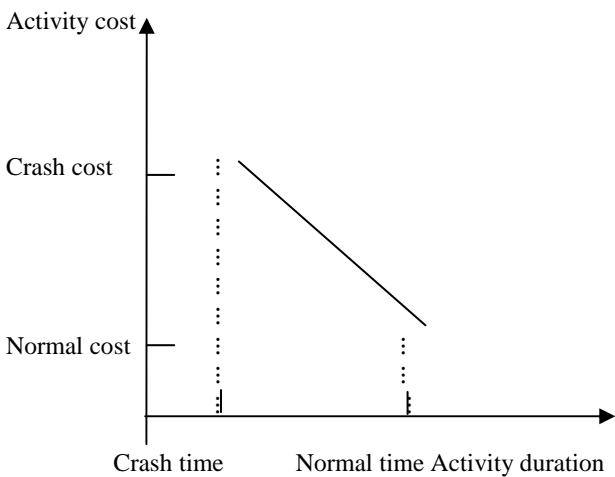
Activity time-cost relationship:

Figure 1. Linear time and cost trade-off for an activity

4. MATHEMATICAL ASPECT

Suppose that the amount of money invested in each critical path activity is c_i where $i = 1, 2, \ldots \ldots \ldots, k$ and k is the number of activities on the critical path. The total extra invested money is M and then it can be written as

$$M = \sum_{i=1}^{k} c_i$$

To minimize expected project duration T_e to T_e^* and project standard deviation σ_r to σ_r^*, we reduce the pessimistic time of critical activities b_i to b_i^*.

Then the mathematical models for new T_e and σ_r stand

$$T_e^* = min.T_e$$

The new expected time for critical path activities,

$$t_i^* = \frac{a_i + 4m_i + b_i^*}{6}$$
$$s/t . b_i^* \le b_i$$

The decreasing of pessimistic time b_i of the critical path activities depends on the increasing of cost c for each activity on critical path. Therefore, it means that the decrease in both expected time and variance of the activities will be a function $\emptyset$ and φ of the additional investment. Then the new pessimistic time and variance of the activities for the critical path activities will be

$$b_i^* = b_i - \emptyset(c_i) \text{ and } \sigma_i^* = \sigma_i - \varphi(c_i)$$
$$\text{with } c_i \ge 0$$

Combining all the constraints from (7), (8), (9) and (10) along with the time function (6), we get the following mathematical model

Time Model

$$T_e^* = min.T_e$$
$$s/t.t_i^* = \frac{a_i + 4m_i + b_i^*}{6}$$
$$b_i^* = b_i - \emptyset(c_i)$$
$$\sigma_i^* = \sigma_i - \varphi(c_i)$$
$$b_i^* \le b_i$$

and $c_i \ge 0$

After reducing the expected time, the probability of the project completion on or before the targeted time increased to Z^*. Then the mathematical form of the objective function can be written as

$$Z^* = max.Z = \frac{T_d - T_e^*}{\sigma_r^*}$$

For getting the increased probability we have to decrease the project duration T_e and to decrease the project standard deviation σ_r, which can be written as

$$T_e^* < T_e$$

and $\sigma_r^* < \sigma_r$

Combining all the constraints from (12), (13), (10) and (8) along with the probability function (11), we get the following mathematical model

Probability Model

$$Z^* = max.Z = \frac{T_d - T_e^*}{\sigma_r^*}$$

s/t $T_e^* < T_e$

$$\sigma_r^* < \sigma_r$$
$$b_i^* \le b_i$$

and $c_i \ge 0$

5. ALGORITHMIC ASPECT

Input: Allocate extra resources in order to reduce the activity time for each critical activity.

Output: Maximum value of the probability of the completion of Project within the deadline.

Scheduling Steps

Step 1: Determine the activities along the critical path.

Step 1.1. Estimate optimistic time (a), most likely (m) and pessimistic time (b) for each activity.

Step 1.2. Calculate expected time for each activity. Use the following formula for activity time duration.

$$t = \frac{a + 4m + b}{6}$$

Step 1.3. Calculate Earliest Start time (ES), Earliest Finisf time (EF), Latest Start time (LS) and Latest Finish time (LF) for each activity by the following rules:

- Forward Pass: Set earliest start time zero for all activities which have no predecessors. Earliest finish time is equal to the sum of earliest start time and activity duration (t) of the activity. To calculate the ES of activities with immediate predecessors set the maximum of the EF of all predecessors to the activity.
- The project completion time (T) is equal to the maximum of the earliest finish time of all activities.
- Backward Pass: The latest finish time of activities with no successors are set equal to the project completion time. Calculate LS of an activity by subtracting the activity duration (t) from it's LF. To calculate LF of activities with successors set the minimum of LS of all successors to the activity.

Step 1.4. Calculate slack for all the activities by one of the following rules:

Rule 1. $Slack = LS - ES$
Rule 2. $Slack = LF - EF$

Step 1.5. Sort the activities with zero slack. These are critical activities and the path containing these activities is critical path.

Step 2: Calculate the project completion time (T_e) by adding the activity times of all critical activities.

Step 3: Determine the Project Completion Probability

Step 3.1. Calculate the variance of all critical activities by the following formula

$$v = \left(\frac{b-a}{6}\right)^2$$

Step 3.2. Sum the variances of all critical activities for the project variance.

Step 3.3. Find the project standard deviation (σ). Use the following formula in this step:

$$\sigma = \sqrt{Project\ variance}$$

Step 3.4. Use the following formula to calculate the probability of the project completion (Z) within the deadline (T_d)

$$Z = \frac{T_d - T_e}{\sigma}$$

Crashing Steps

Step 4: Fix an amount of resources and distribute it to all the critical activities.

Step 5: Reduce the pessimistic time (b) for all critical activities.

Step 6: Recall the scheduling steps to calculate a new value of Z.

Step 7: Continue the process until Z become maximum.

6. NUMERICAL ASPECT

We consider a hypothetical project with 12 activities. Optimistic time (a), Most likely time (m), Pessimistic time (b) and immediate predecessor of each activity of the project is given in the following table:

Table 1. a, b, m and immediate predecessors of the 12 activities.

Activity	a	m	b	Immediate Predecessors
ac_1	8	10	12	-
ac_2	6	7	9	-
ac_3	3	3	4	-
ac_4	10	20	30	ac_1
ac_5	6	7	8	ac_3
ac_6	9	10	11	ac_2, ac_4, ac_5
ac_7	6	7	10	ac_2, ac_4, ac_5
ac_8	14	15	16	ac_6
ac_9	10	11	13	ac_6
ac_{10}	6	7	8	ac_7, ac_8
ac_{11}	4	7	8	ac_9, ac_{10}
ac_{12}	1	2	4	ac_7, ac_8

We use PERT probabilistic method to calculate expected time (t), earliest start time (ES), latest start time (LS) and variance (v) of each activity and tabulated in the table 2.

Table 2. Values of t, ES, LS and variance of 12 activities using PERT method.

Activity	Expected time t	Earliest Start time ES	Latest Start time LS	Slack $LS - ES$	Variance v
ac_1	10	0	0	0	0.44
ac_2	7.17	0	22.83	22.83	0.25
ac_3	3.17	0	19.83	19.83	0.03
ac_4	20	10	10	0	11.11
ac_5	7	3.17	23	19.83	0.11
ac_6	10	30	30	0	0.11
ac_7	7.33	30	47.67	17.67	0.44
ac_8	15	40	40	0	0.11
ac_9	11.17	40	50.83	10.83	0.25
ac_{10}	7	55	55	0	0.11
ac_{11}	6.67	62	62	0	0.44
ac_{12}	2.17	66.5	68.67	2.17	0.25

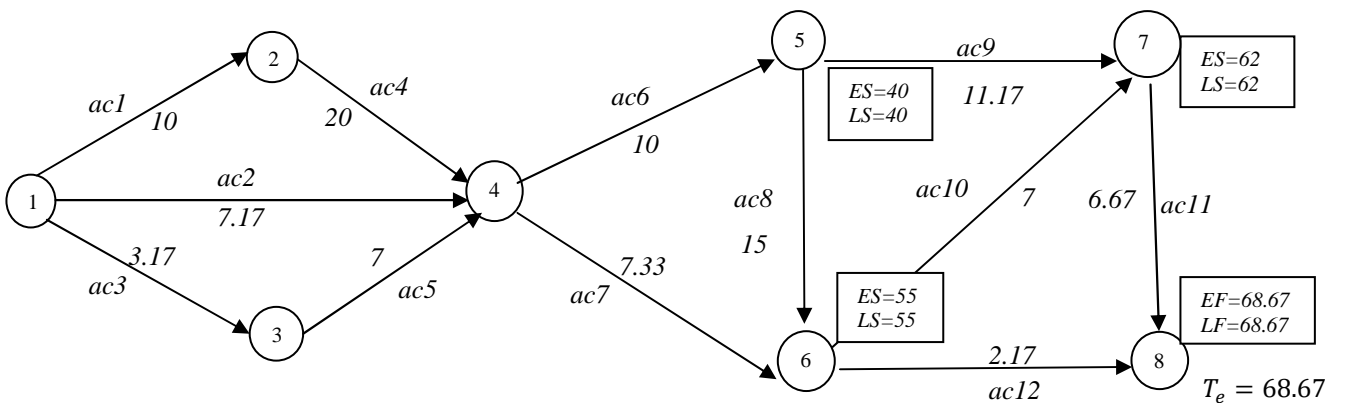Project network showing the critical path:

Figure 2. Critical path on PERT network

The activities with slack 0 are on the critical path of the network which is indicated with bold line.

Expected project completion date

$$T_e = 10 + 20 + 10 + 15 + 7 + 6.67 = 68.67 \text{ days.}$$

It means that, there is a 50% chance that the entire project will be completed in less than the expected 68.67 days and a 50% chance that it will exceed 68.67 days.

Project variance

$$\sigma_r = \sqrt{0.44 + 11.11 + 0.11 + 0.11 + +0.11 + 0.44}$$
$$= \sqrt{12.32} = 3.51$$

Probability to meet the normal time:

To find the probability that the project will be completed on or before 70 days deadline, we need to determine the value of Z

$$Z = \frac{T_d - T_e}{\sigma_r} = \frac{70 - 68.67}{3.51} = 0.38$$

From the standard normal table, we find the probability 0.6480. Thus there is a 64.8% chance that the project will be completed in 70 days.

Probability to meet the targeted time:

To find the probability that the project will be finished on or before 65 days deadline without any extra investment, the value of Z is

$$Z = \frac{T_d - T_e}{\sigma_r} = \frac{65 - 68.67}{3.51} = -1.046$$

From the standard normal table, we find the probability 0.1469. Thus there is only 14.69% chance that the project will be completed in 65 days within the same budget.

According to the new approach presented in this paper, we invest approximately Tk.10000, Tk.20000 and Tk.30000 extra budgets for the critical path activities to show how the probability of project completion increases. As a result we reduce b to b^* for each critical activity, shown in following tables:

Table 3. Estimation of b, t and v after increased the budget amount M=Tk.10000.

Critical activity	a	m	b^*	t^*	v^*
ac_1	8	10	11.5	9.92	0.34
ac_4	10	20	28	19.67	9
ac_6	9	10	10.75	9.96	0.09
ac_8	14	15	15.5	14.92	0.06
ac_{10}	6	7	7.75	6.96	0.09
ac_{11}	4	7	7.75	6.63	0.39

From the Table 3,

$$T_e^* = min. T_e = 68.06$$
$$\text{and } \sigma_r^* = \sqrt{9.67} = 3.16$$
$$Z^* = max. Z = \frac{T_d - T_e^*}{\sigma_r^*} = \frac{65 - 68.06}{3.16} = -0.97$$

Table 4. Estimation of *b, t* and *v* after increased the budget amount M=Tk.20000.

Critical activity	a	m	b^*	t^*	v^*
ac_1	8	10	11	9.83	0.25
ac_4	10	20	26	19.33	7.11
ac_6	9	10	10.5	9.92	0.06
ac_8	14	15	15.25	14.86	0.04
ac_{10}	6	7	7.5	6.92	0.06
ac_{11}	4	7	7.5	6.58	0.34

From the above Table 4,

$$T_e^* = min.T_e = 67.44$$
$$\text{and } \sigma_r^* = \sqrt{7.86} = 2.80$$
$$Z^* = max.Z = \frac{T_d - T_e^*}{\sigma_r^*} = \frac{65 - 67.44}{2.80} = -0.87$$

Table 5. Estimation of *b, t* and *v* after increased the budget amount M=Tk.30000.

Critical activity	a	m	b^*	t^*	v^*
ac_1	8	10	10.5	9.75	0.17
ac_4	10	20	24	19	5.44
ac_6	9	10	10.25	9.88	0.04
ac_8	14	15	15	14.83	0.03
ac_{10}	6	7	7.25	6.88	0.04
ac_{11}	4	7	7.25	6.54	0.29

From the Table 5,

$$T_e^* = min.T_e = 66.88$$
$$\text{And } \sigma_r^* = \sqrt{6.01} = 2.45$$
$$Z^* = max.Z = \frac{T_d - T_e^*}{\sigma_r^*} = \frac{65 - 66.88}{2.45} = -0.77$$

Increasing of the probability that the project will be finished within 65 days is shown in the following table:

Table 6. The probability that the project will be completed in 65 days when

$$M = \{0, Tk.\,10000, Tk.\,20000, Tk.\,30000\}.$$

No.	M	σ_r^*	T_d	T_e^*	Z	%
1	0	3.51		68.67	-1.046	14.78
2	Tk. 10000	3.16		68.06	-0.97	16.60
3	Tk. 20000	2.80	65	67.44	-0.87	19.22
4	Tk. 30000	2.45		66.88	-0.77	22.06

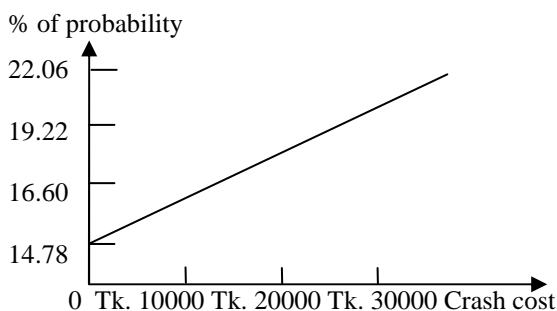The results are shown in the following graph:

Figure 3. A typical project completion probability-cost graph for an activity

7. CONCLUSIONS

We estimated the probability that the project will be completed in the targeted time for different cases. If we invest the extra budgets approximately Tk.10000, Tk.20000 and Tk.30000 for the activities along the critical path, the expected time of the project decreases to 68.06 days, 67.44 days and 66.88 days respectively. And the probability that the project will be finished within the targeted 65 days increases from 14.69% to 16.60%, 19.22% and 22.06%.The project completion probability within the targeted date can be expressed as the following composite relations

$$Z = f(T), T = g(b) \text{ and } b = h(c)$$

Where, Z is the number of standard deviations, T is the project completion time, b is the pessimistic time for each critical path activity and c is the extra cost invested for each such activity.

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